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Padé Approximants as Limits of Rational Functions of Best Approximation, Real Domain*

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The Padé approximant to a given function $f(x)$ is the *rational function* $P_{n\nu}(x)$ *of type* (n, ν):

$$\frac{s_0 + s_1 x + \cdots + s_n x^n}{t_0 + t_1 x + \cdots + t_\nu x^\nu}, \qquad \sum |t_k| \neq 0,$$

with contact of the highest order at the origin to $f(x)$ of class $C^{(n+\nu+1)}[0, 1]$:

$$f(x) \equiv a_0 + a_1 x + \cdots + a_{n+\nu} x^{n+\nu} + O(x^{n+\nu+1}), \qquad a_0 \neq 0. \tag{1}$$

It is shown in [1] that provided a certain determinant of the a_k is not zero, the rational function $R_{n,\nu}(\epsilon, x)$ of type (n, ν) of best approximation to $f(x)$ (assumed analytic) on the disc $|x| \leqslant \epsilon$ as $\epsilon \to 0$ approaches as a limit the function $P_{n\nu}(x)$ on any closed set within which $P_{n\nu}(x)$ is analytic. The object of the present note is to prove the analogous theorem in the real domain, a hitherto open question suggested to me by Dr. Oved Shisha.

The method of Padé is as follows. With $f(x)$ given by (1) we need to determine

$$P_{n\nu}(x) \equiv \frac{s_0 + s_1 x + \cdots + s_n x^n}{t_0 + t_1 x + \cdots + t_\nu x^\nu} \equiv \sum_{k=0}^{n+\nu} a_k x^k + O(x^{n+\nu+1}). \tag{2}$$

As Padé shows, the determination of the s_i and t_i is equivalent to the determination of $t_0, t_1, \ldots, t_\nu$ and the d_i, where we set

$$\sum_{j=0}^{n+\nu} a_j x^j \cdot \sum_{k=0}^{\nu} t_k x^k \equiv \sum_{i=0}^{n+\nu} d_i x^i + O(x^{n+\nu+1}) \tag{3}$$

and where $d_{n+1} = d_{n+2} = \cdots = d_{n+\nu} = 0$. This determination is in turn

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equivalent to the solution for the numbers $t_0, t_1, \ldots, t_\nu$ from the two sets of equations

$$\left.\begin{aligned} a_0 t_0 &= d_0 = s_0, \\ a_1 t_0 + a_0 t_1 &= d_1 = s_1, \\ &\cdots\cdots\cdots \\ a_n t_0 + a_{n-1} t_1 + \cdots + a_{n-\nu} t_\nu &= d_n = s_n; \end{aligned}\right\} \tag{4}$$

$$\left.\begin{aligned} a_{n+1} t_0 + a_n t_1 + \cdots + a_{n-\nu+1} t_\nu &= d_{n+1} = 0, \\ a_{n+2} t_0 + a_{n+1} t_1 + \cdots + a_{n-\nu+2} t_\nu &= d_{n+2} = 0, \\ &\cdots\cdots\cdots \\ a_{n+\nu} t_0 + a_{n+\nu-1} t_1 + \cdots + a_n t_\nu &= d_{n+\nu} = 0. \end{aligned}\right\} \tag{5}$$

Equations (4) and (5) are written for the case $n \geqslant \nu$; in the contrary case the numbers a_i with negative subscripts are to be taken as zero.

We shall treat $R_{n\nu}(\epsilon, x)$ formally by equations precisely similar to (3), (4), and (5), where $f(x)$ is still given by (1), but except that $R_{n\nu}(\epsilon, x)$ of type (n, ν) is now determined by its property of best approximation to $f(x)$ on the segment δ: $[0, \epsilon]$; we have

$$R_{n\nu}(\epsilon, x) = \frac{u_0 + \cdots + u_n x^n}{v_0 + \cdots + v_\nu x^\nu} \equiv \sum_{k=0}^{n+\nu} b_k x^k + O(x^{n+\nu+1}), \tag{6}$$

where the coefficients depend on ϵ.

These coefficients $b_0, b_1, \ldots, b_\nu$ are related to the u_i and the v_i by the sets of equations

$$\sum_{j=0}^{n+\nu} b_j x^j \cdot \sum_{k=0}^{\nu} v_k x^k \equiv \sum_{i=0}^{n+\nu} u_i x^i + O(x^{n+\nu+1}), \tag{7}$$

$$\left.\begin{aligned} b_0 v_0 &= u_0, \\ b_1 v_0 + b_0 v_1 &= u_1, \\ &\cdots\cdots\cdots \\ b_n v_0 + b_{n-1} v_0 + \cdots + b_{n-\nu} v_\nu &= u_n; \end{aligned}\right\} \tag{8}$$

$$\left.\begin{aligned} b_{n+1} v_0 + b_n v_1 + \cdots + b_{n-\nu+1} v_\nu &= 0, \\ b_{n+2} v_0 + b_{n+1} v_1 + \cdots + b_{n-\nu+2} v_\nu &= 0, \\ &\cdots\cdots\cdots \\ b_{n+\nu} v_0 + b_{n+\nu-1} v_0 + \cdots + b_n v_\nu &= 0; \end{aligned}\right\} \tag{9}$$

these equations too are written for $n \geqslant \nu$, but for $\nu > n$ we consider all b_i with negative subscripts to be zero. Of course, equations (9) can perhaps be continued, but that is not necessary for our present purposes.

We shall prove our principal result:

THEOREM 1. *Let the function*

$$f(x) \equiv a_0 + a_1 x + \cdots + a_{n+\nu} x^{n+\nu} + O(x^{n+\nu+1}), \qquad a_0 \neq 0,$$

of class $C^{(n+\nu+1)}[0, 1]$ *or of some class* $C^{(n+\nu+1)}[0, \epsilon]$, $\epsilon > 0$, *for* $\epsilon\,(> 0)$ *sufficiently small, and fixed n and* ν, *and let* $R_{n\nu}(\epsilon, x)$ *denote the function of type* (n, ν) *of best approximation to* $f(x)$ *in the (uniform) sense of Tchebycheff on the interval* δ: $0 \leqslant x \leqslant \epsilon$. *Suppose we have*

$$\Delta_{n-1,\nu-1} = \begin{vmatrix} a_n & a_{n-1} & \cdots & a_{n-\nu+1} \\ a_{n+1} & a_n & \cdots & a_{n-\nu+2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+\nu-1} & a_{n+\nu} & \cdots & a_n \end{vmatrix} \neq 0; \tag{10}$$

then as ϵ *approaches zero* $R_{n\nu}(\epsilon, x)$ *approaches the Padé function* $P_{n\nu}(x)$ *of* (2) *on any clased set where* $P_{n\nu}(x)$ *is analytic.*

Both $P_{n\nu}(x)$ and $R_{n\nu}(\epsilon, x)$ are of type (n, ν), so by the extremal property of $R_{n\nu}(\epsilon, x)$ we have

$$[\max |f(x) - R_{n\nu}(\epsilon, x)|, x \text{ on } \delta] \leqslant [\max |f(x) - P_{n\nu}(x)|, x \text{ on } \delta] \tag{11}$$

and by Taylor's theorem with remainder, for x on δ for the (Tchebycheff) norms

$$\|f(x) - R_{n\nu}(\epsilon, x)\|_\delta \leqslant \|f(x) - P_{n\nu}(x)\|_\delta \leqslant M\epsilon^{n+\nu+1},$$

where $M = \max[|f^{n+\nu+1}(x) - P_{n\nu}^{n+\nu+1}(x)|, x \text{ on } \delta]/(n + \nu + 1)!$. Then we also have

$$\| P_{n\nu}(x) - R_{n\nu}(\epsilon, x)\|_\delta \leqslant 2M\epsilon^{n+\nu+1}. \tag{12}$$

In other symbols we have

$$\left\| \sum_{k=0}^{n+\nu} (a_k - b_k)\, x^k \right\|_\delta \leqslant 2M\epsilon^{n+\nu+1}. \tag{13}$$

It now follows from Lemma 2 proved below that as a consequence of (13)

$$| a_k - b_k | = O(\epsilon) \qquad \text{for} \quad k = 0, 1, \ldots, n + \nu. \tag{14}$$

The conclusion of Theorem 1 follows, from the fact that these $n + \nu + 1$ coefficients b_k are "near" the corresponding a_k, the equations (9) and (8) for the u_k and v_k are "near" the equations (5) and (4) for the s_k and t_k respectively, and hence their unique solutions u_k and v_k are "near" the s_k and t_k. To be more explicit, let us adjoin to the system (9) the equation $v_0 = v$, where v is a multiplicative parameter. We now have $\nu + 1$ equations with $\nu + 1$ unknowns $v_0, v_1, \ldots, v_\nu$; for ϵ sufficiently small the determinant of the system is different from zero, by (10) and (14). The numbers $v_1, v_2, \ldots, v_\nu$ and $u_0, u_1, \ldots, u_n$ are then uniquely determined by (8) from $b_0, b_1, \ldots, b_{n+\nu}$ in terms of the parameter v. Of course equation (6) determines the u_j and v_j from the b_k merely to within a multiplicative constant; we shall consider such determination as determining the u_j and v_j uniquely. We adjoin similarly the equation $t_0 = v$ to the system (5), so (5) determines $t_0, t_1, \ldots, t_\nu$, and (4) determines the numbers $s_0, s_1, \ldots, s_n$ uniquely in terms of the multiplicative parameter v. The coefficients u_j and v_j in (6) can be made to differ by as small an amount as we please from the corresponding coefficients s_j and t_j in (2), merely by choosing ϵ sufficiently small, and we may choose $v_0 = t_0 = v = 1$; the conclusion of Theorem 1 follows.

It remains to establish two lemmas.

LEMMA 1. *With the hypothesis* $P(x) \equiv \sum_{k=0}^{N} A_k x^k$, $|P(x)| \leqslant Q$ *for* $0 \leqslant x \leqslant 1$, *we have also* $|A_j| \leqslant CQ$, *where* C *is independent of* Q.

Let the Tchebycheff polynomials $t_0(x), t_1(x), \ldots, t_N(x)$ of respective degrees $0, 1, \ldots, N$ be normal and orthogonal on $[0, 1]$. Then we have

$$P(x) \equiv \sum_{k=0}^{N} B_k t_k(x), \qquad B_k = \int_0^1 P(x)\, t_k(x)\, dx, \tag{15}$$

and Bessel's inequality

$$\sum_{k=0}^{N} B_k^2 \leqslant \int_0^1 [P(x)]^2\, dx \leqslant Q^2. \tag{16}$$

However, $t_k(x)$ can be expressed uniquely in terms of the set $\{x^j, j = 0, 1, \ldots, k\}$:

$$t_k(x) = C_{k0} + C_{k1}x + \cdots + C_{kk}x^k,$$

where the numerical coefficients C_{kj} are well-known. Then we have

$$P(x) \equiv \sum_{k=0}^{N} B_k(C_{k0} + C_{k1}x + \cdots + C_{kk}x)^N$$

$$\equiv \sum_{k=0}^{N} B_k C_{k0} + \sum_{k=1}^{N} B_k C_{k1} x + \cdots + \sum_{k=N}^{N} B_k C_{kk} x^N.$$

Moreover, since the powers of x are linearly independent on [0, 1], we may write

$$A_0 = \sum_{k=0}^{N} B_k C_{k0}\,, \qquad A_1 = \sum_{k=1}^{N} B_k C_{k1}\,,\ldots, \qquad A_N = \sum_{k=N}^{N} B_k C_{kk}\,.$$

By the Cauchy-Schwarz inequality we have

$$\sum_{k=0}^{N} A_k^2 \leqslant \sum_{k=0}^{N} B_k^2 \left[\sum_{k=0}^{N} C_{k0}^2 + \sum_{k=1}^{N} C_{k1}^2 + \cdots + \sum_{k=N}^{N} C_{kk}^2\right]$$

$$\leqslant Q^2 \left[\sum_{k=0}^{N} C_{k0}^2 + \sum_{k=1}^{N} C_{k1}^2 + \cdots + \sum_{k=N}^{N} C_{kk}^2\right], \tag{17}$$

which gives the conclusion of Lemma 1.

LEMMA 2. *With the hypothesis* $P(t) \equiv \sum_{k=0}^{N} A_k' t^k$, $|P(t)| \leqslant Q_1$ *for* $0 \leqslant t \leqslant r$, *we have also* $|A_j'|\, r^j \leqslant C' Q_1$, *where* C' *is independent of* Q_1 *and* r.

We set here $t = rx$, $dt = r\,dx$, $x = t/r$; then we study $P(rx)$ on $0 \leqslant x \leqslant 1$, whence

$$\int_0^1 [P(rx)]^2\, dx \leqslant Q_1^2, \qquad P(rx) \equiv \sum_{k=0}^{N} (A_k' r^k)\, x^k.$$

A conclusion of Lemma 1 may be taken as (17), now in the form

$$\sum_{k=0}^{N} A_k^2 \leqslant CQ^2, \qquad A_j = A_j' \cdot r^j, \tag{18}$$

where we have in the notation of Lemma 2 ($0 \leqslant x \leqslant 1$)

$$\sum_{k=0}^{N} (A_k' r^k)^2 / C \leqslant \int_0^1 [P(rx)]^2\, dx \leqslant Q_1^2. \tag{19}$$

We derive from (19) the conclusion of Lemma 1:

$$|A_j'|\, r^j \leqslant C' \cdot Q_1\,.$$

In the proof of Theorem 1, we have (13), which may be taken in the form ($n + \nu$ constant)

$$\left| \sum_{k=0}^{n+\nu} (a_k - b_k)\, x^k \right| \leqslant 2M\epsilon^{n+\nu+1}, \qquad \text{for} \quad 0 \leqslant x \leqslant \epsilon.$$

There follows by Lemma 2

$$| a_k - b_k | \epsilon^k \leqslant 2MC'\epsilon^{n+\nu+1}, \qquad k = 0, 1, 2,..., n + \nu,$$

which yields (14) and thus completes the proof of Theorem 1.

It may be noticed that the conclusion of Theorem 1 follows from (11) without explicit extremal assumptions on $R_{n\nu}(x, \epsilon)$.

The problems of [1, $_\nu$4] and of the present Theorem 1 were mentioned in [2] regarding the polynomial $P_{n0}(x)$ as the limit of the polynomial $R_{n0}(x, \epsilon)$ as $\epsilon \to 0$, both in the real case and complex case, but without the firm conclusions on $P_{n\nu}(x)$ and $R_{n\nu}(x, \epsilon)$ established in [1] and here.

REFERENCES

1. J. L. WALSH, Padé approximants as limits of rational functions of best approximation, *J. Math. Mech.* **13** (1964), 305–312.
2. J. L. WALSH, On approximation to an analytic function by rational functions of best approximation, *Math. Z.* **38** (1934), 163–176.